

# Eigenvalue Problem for Schrödinger Operators and Time-Dependent Harmonic Oscillator

Ali Mostafazadeh\*

Theoretical Physics Institute, University of Alberta,  
Edmonton, Alberta T6G 2J1, Canada,

and

Department of Mathematics, College of Arts and Sciences,  
Koç University, Istinye, 80860 Istanbul, Turkey<sup>†</sup>

June 1997

## Abstract

It is shown that the eigenvalue problem for the Hamiltonians of the standard form,  $H = p^2/(2m) + V(x)$ , is equivalent to the classical dynamical equation for certain harmonic oscillators with time-dependent frequency. This is another indication of the central role played by time-dependent harmonic oscillators in quantum mechanics. The utility of the known results for eigenvalue problem in the solution of the dynamical equations of a class of time-dependent harmonic oscillators is also pointed out.

---

\*E-mail: alimos@phys.ualberta.ca

<sup>†</sup>Address after July 1, 1997

Recently there has been a growing interest in the study of the quantum dynamics, i.e., solution of the Schrödinger equation

$$H\psi = i\hbar \frac{d\psi}{dt} , \quad (1)$$

for time-dependent harmonic oscillator [1],

$$H = \frac{p^2}{2M(t)} + \frac{1}{2} M(t)\omega(t)^2 x^2 , \quad (2)$$

and its generalizations [2]. The basic idea used in these studies is the invariant method of Lewis and Riesenfeld [3]. Although the results of Refs. [1, 2] have direct relevance for the construction of the squeezed states which have potential physical applications, contrary to the claims made by the authors, they do not yield exact solution of the Schrödinger equation (1). In reality, what is being done [1] is to show that the general solution of the Schrödinger equation can be expressed in terms of the solutions of the classical equation of motion. This is a second order differential equation with variable coefficients whose exact solution is not known. The purpose of this article is to show that an exact solution of the dynamical equation for time-dependent harmonic oscillator includes as a special case the solution for the eigenvalue problem for arbitrary time-independent Hamiltonians of the standard form  $H = p^2/(2m) + V(x)$ , i.e.,

$$\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi_E = E \psi_E . \quad (3)$$

It is well-known that the eigenvalue problem (3) can be viewed as a variational problem for the energy [4]

$$\begin{aligned} \mathcal{E}[\psi] &:= \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{1}{(N[\psi])^2} \int dx \, \psi^* \left( -\frac{\hbar^2}{2m} \psi'' + V\psi \right) , \\ (N[\psi])^2 &:= \int dx \, \psi^* \psi , \quad \psi'' = \frac{d^2 \psi}{dx^2} , \end{aligned} \quad (4)$$

i.e., Eq. (3) can be easily shown to be equivalent to

$$\frac{\delta \mathcal{E}[\psi]}{\delta \psi^*(x)} = 0 , \quad (5)$$

where  $\delta/\delta\psi^*(x)$  denotes a functional derivative. Performing a simple integration by parts, one can formulate the corresponding variational problem as that of the action functional  $S = \int L \, dx$ ,

where  $L$  is the Lagrangian

$$L := \frac{1}{2(N[\psi])^2} \left( \psi^{*'} \psi' + \frac{mV(x)}{\hbar^2} \psi^* \psi \right) , \quad (6)$$

and a prime means  $d/dx$ . Note that for the dynamical system described by  $L$ ,  $\psi$  and  $x$  play the role of the ‘position’ and ‘time’, respectively. Unfortunately, due to the presence of  $N$  on the right-hand side of (6),  $L$  is not in the standard form.

In order to remedy this problem, let us introduce the normalized wave functions  $\Psi := \psi/N[\psi]$ . Then,  $L$  is equivalent to the Lagrangian

$$L' := \frac{1}{2} \left[ \Psi^{*'} \Psi' + \frac{mV(x)}{\hbar^2} \Psi^* \Psi \right] + \lambda \left( 1 - \int dx \Psi^* \Psi \right) , \quad (7)$$

where  $\lambda$  is a Lagrange multiplier enforcing the normalization constraint.

Switching to real variables one can easily identify (7) as the Lagrangian for a ‘time’-dependent two-dimensional circular harmonic oscillator [5] with coordinates  $(q_1, q_2)$  and frequency  $\omega(x)$  given by

$$\Psi =: q_1 + iq_2 , \quad \omega^2 = -\frac{mV(x)}{\hbar^2} .$$

In fact, without loss of generality, one can assume that  $\psi$  and consequently  $\Psi$  are real.<sup>1</sup> In this case, one has

$$L' = \frac{1}{2} \left[ \Psi'^2 - \omega(x)^2 \Psi^2 \right] + \lambda \left( 1 - \int dx \Psi^2 \right) , \quad (8)$$

This is precisely the Lagrangian for a one-dimensional harmonic oscillator of unit mass and ‘time’-dependent frequency  $\omega(x)$  whose position  $\Psi$  is constrained to satisfy  $\Phi := 1 - \int \Psi^2 dx = 0$ . The constraint  $\Phi = 0$  is a rather unusual global constraint demonstrating the global character of the eigenvalue problem.

There is another way of relating the eigenvalue problem (3) to time-dependent harmonic oscillators which separates the local and global features of the eigenvalue problem. In order to outline this approach, let us define the following one-parameter family of the actions  $S_E[\Psi] = \int L_E(\Psi, \Psi', x) dx$  with

$$L_E := \frac{1}{2} \left[ \Psi^{*'} \Psi' - \Omega_E^2(x) \Psi^* \Psi \right] , \quad E \in \mathbb{R} , \quad (9)$$

---

<sup>1</sup>This is because the Schrödinger operator appearing in Eq. (3) is real and linear.

and  $\Omega_E^2 := m[E - V(x)]/\hbar^2$ . One can easily show that the classical equation of motion corresponding to  $S_E$  is identical with Eq. (3). However, not for every  $E$  are the solutions square integrable. The requirement of the square integrability of the classical solution  $\Psi_E$  is equivalent to the condition

$$N[\Psi_E] = 1. \quad (10)$$

Hence, the eigenvalues  $E$  corresponds to those solutions, i.e., classical trajectories  $\Psi_E = \Psi_E(x)$ , which satisfy (10).

Again, one can see that  $L_E$  is the Lagrangian for a time-dependent two-dimensional circular harmonic oscillator. Choosing,  $\Psi$  real, one obtains a one-dimensional harmonic oscillator with time-dependent frequency  $\Omega_E$ , i.e.,

$$L_E = \frac{1}{2} [\Psi'^2 - \Omega_E^2(x) \Psi^2], \quad \Omega_E^2 := \frac{m[E - V(x)]}{\hbar^2}. \quad (11)$$

Therefore, if one could solve the classical dynamics of such oscillators, one would be able to reduce the eigenvalue equation (3) to an algebraic equation, namely Eq. (10).

This observation clearly demonstrates the central role time-dependent harmonic oscillators play in spectral analysis of one-dimensional Schrödinger operators. It also indicates the degree of difficulty of the exact solution of the classical and in view of Refs. [1] the quantum dynamics of time-dependent harmonic oscillators.

Another interesting observation is to use the knowledge about the known solutions of the eigenvalue problem (3) to construct solutions of the classical dynamics of certain time-dependent oscillators. In order to do this, one needs to select a potential  $V$  for which some or all of the eigenfunctions  $\Psi_E$  are known. These would directly yield certain exact solutions of the classical equation of motion for (11). For example, let  $V = \omega_0^2 x^2/2$  for some real constant  $\omega_0$ , then  $E = (n + 1/2)\hbar\omega_0$  and  $\Psi_E$  are the known normalized eigenfunctions of the one-dimensional time-independent harmonic oscillator. In view of the preceding discussion,  $\Psi_E$  are also exact solutions for the classical dynamical equations for ‘time’-dependent harmonic oscillators with frequency

$$\Omega_n(x) = \sqrt{\frac{m\omega_0}{\hbar} \left( n + \frac{1}{2} - \frac{\omega_0 x^2}{2\hbar} \right)}, \quad n = 0, 1, 2, \dots, \quad (12)$$

and unit mass. Note that the frequency  $\Omega_n$  become imaginary for the classically forbidden region.

Similarly, one can use the results for other exactly solvable eigenvalue problems [5] to obtain exact solutions for the classical dynamical equations of the corresponding time-dependent harmonic oscillators. In view of the results of Ref. [1] this would also yield the exact solution of the Schrödinger equation for these oscillators.

The same approach may also be applied for higher-dimensional eigenvalue problems. In this case, one can show the equivalence of the eigenvalue equation and the classical field equation for certain non-interacting Euclidean scalar field theories with variable mass.

In conclusion, the time-dependent harmonic oscillators are shown to play a most prominent role in the spectral theory of Schrödinger operators. The relationship between these two apparently distinct subjects can be used to yield exact solution for the classical dynamics of a class of time-dependent oscillators. The implications of this observation in the construction of squeezed states as outlined in Ref. [1] awaits further investigation.

## References

- [1] J-Y. Ji, J. K. Kim, and S. P. Kim, Phys. Rev. **A 51**, 4268 (1995);  
Y-Z. Lai, J-Q. Liang, H. J. W. Müller-Kirsten, and J-G. Zhou, J. Phys. A: Math. Gen. **29**, 1773 (1996);  
J-Y. Ji and K-S. Soh, “Quantum theory of a time-dependent harmonic oscillator in the pilot-wave theory,” Seoul National Uni. preprint no: SNUTP 97-003, quant-ph/9701002.
- [2] H-C. Kim, M-H. Lee, J-Y. Ji, and J. K. Kim, Phys. Rev. **53**, 3767 (1996);  
M-H. Lee, H-C. Kim and J-Y. Ji, “Exact wave functions and geometric phases of a generalized driven oscillator,” Korea Advanced Institute of Science and Technology preprint no: KAIST-CHEP-96/8, quant-ph/9701033.
- [3] H. R. Lewis, Jr., and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
- [4] P. Blanchard and E. Brüning, *Variational Methods in Mathematical Physics* (Springer-Verlag, Berlin, 1992).
- [5] S. Flügge, *Practical Quantum Mechanics I* (Springer-Verlag, Berlin, 1971).